The Associated Metric for a Particle in a Quantum Energy Level.

E. Atzmon[*]

Raymond and Beverly Sackler Faculty of Exact Sciences,

School of Physics and Astronomy.

Tel - Aviv University.

February 1, 2008

Abstract

We show that the probabilistic distribution over the space in the spectator world, can be associated via noncommutative geometry (with some modifications) to a metric in which the particle lives. According to this geometrical view, the metric in the particle world is "contracted" or "stretched" in an inverse proportion to the probability distribution.

[*]atzmon@post.tau.ac.il

1 Prelude

It is well known that for many physical quantum potentials, the spectrum of the Hamiltonian is a discrete set (i.e. the set of energy eigenvalues E_n is a countable set) associated with a separable Hilbert space of eigenstates ψ_n.[1] It is also well known that in Quantum Mechanics, $|\psi(x)|^2$ defines a probability distribution over coordinate space, in the observer world. Using the formula for distance in noncommutative geometry [1], we were recently able to show that one might then also associate the relevant metric with particle energy levels. In noncommutative geometry, one uses the spectral triple (A, H, D) , where A is the algebra of functions on the space (which according to the Gelfand-Naimark theorem [2] is fully characteristic of the space), H is a Hilbert space, and D is a Dirac operator, which acts as a linear operator on that Hilbert space. The Dirac operator is a self-adjoint operator, with compact resolvent. In recent years,noncommutative geometry [3] has provided the means of further extending the geometric approach to gauge theories with spontaneous symmetry breakdown. In this study, we point out yet another possible application, namely the definition of a metric associated with quantum spectra. At this stage we limit our treatment to its evaluation and a discussion of its consistency with the probabilistic intuitive picture.

The distance formula in noncommutative geometry is:

$$d(a,b) = \sup_f \left\{ |f(a) - f(b)| \ : \ f \in A, \ \| \, [\, D, f \,] \, \| \leq 1 \right\} \tag{1}$$

where $a, b \in X$, $f \in A$, A is the algebra of functions on X, and the norm on the r.h.s is the norm of operators in H. Both the noncommutative geometrical and the classical (i.e. the infimum among all the paths which connect a and b) definitions for distance give the same result when the base space is a Riemannian manifold of genus zero. However the n.c.g definition has the advantage of being applicable to discrete spaces too, on which the concept of a "path" is not well defined.

---

[1]Over compact spaces the spectrum of the Hamiltonian is separable.

The main advantage of the distance formula, eq.(1), is that it involves not just the algebra of functions defined on the space, but also the Hilbert space, which is usually associated with Quantum Mechanics. Had the geometry's defining triple not involved a Hilbert space of states, the algebra of functions would have been applied to a fixed manifold (i.e. a topological manifold, without any dynamics). However, once we have a Hamiltonian H, its spectrum is associcated with a Hilbert space of eigenstates. The eigenstates essentially fix the probability distribution of the particle's position in space.[2] The base space can thus now be considered as a dynamical space, i.e., a space depending on the quantum state (and therefore also on the energy level). This can be seen in every scattering process, e.g. note how visible light is scattered on drops forming a rainbow, due to spectral decomposition, so that each frequency has it own geodesic line. One might well claim that there is an interplay between the fixed base space, and the Hilbert space.

In the following section we will give a metric interpretation, as "seen" by the particle, in its own world, which is consistent with the distributive aspects assigned to the wave function, in quantum mechanics for the observable world.

2 The Quantum Mechanical Distance Formula

Let assume for simplicity that we are giving a one dimensional Hamiltonian system which found in its lowest eigenstate ψ. While using eq.(1) over a smooth manifold, one can prove that:

$$\sup_f |f(a) - f(b)| = \sup_f \left| \int_a^b |\nabla f|\, ds \right| = \left| \int_a^b \sup_f |\nabla f|\, ds \right| \tag{2}$$

---

[2]In Quantum Mechanics the probability distribution of the particle's position in space applies to a single particle, and not only as a statistical result due to large number of measurements. This is just as in the two slits experiment, where a single particle passes through both slits.

3

(where a, b not necessarily ordered).

The norm in eq.(1) usually defined as follows:

$$\| \, [\, D, f \,] \, \| = \sup \{ \| \, [\, D, f \,] \, \psi \, \| \quad \text{where} \quad \| \psi \| \leq 1 \} \tag{3}$$

However, we suggest to modify the norm condition in eq.(1) to be interpreted as a condition on the expectation value of $\| [\, D, f \,] \|$ in the quantum state ψ - that is:

$$1 \geq \langle \, \| [\, D, f \,] \| \, \rangle = \langle \psi \, \| [\, D, f \,] \| \, \psi \rangle = \int_v \psi^*(x) \, \| [\, D, f \,] \| \, \psi(x) dx \tag{4}$$

where v is the entire base space, and we are assuming also that ψ is normalized:

$$\langle \psi | \psi \rangle = \int_v \psi^*(x) \psi(x) dx = 1 \tag{5}$$

Therefore, in order to be consistent with both eq.(5) and the norm condition in eq.(1) replaced by the new "norm" condition in eq.(4), it is sufficient to require the following local condition:

$$\| [\, D, f \,] \| \, |_x = |\nabla f| \, |_x \leq \frac{1}{Vol(v) \psi^*(x) \psi(x)} \tag{6}$$

where $Vol(v)$ is the world volume that we assume to be finite. The idea to replace the norm condition in eq.(1) by eq.(4), follows from the fact that in Q.M. one considers only expectation values as observables. This would enable a quantum state to affect the observed metric as will be shown immediately. By applying eq.(6) to eq.(2), one should use the equality in eq.(6). So if $\forall x \in X, \exists \psi(x) \neq 0$ (except on the boundaries), which is always the situation at the first energy level, the distance formula eq.(1) becomes:

$$d(a,b) = \frac{1}{Vol(v)} \left| \int_a^b \frac{1}{\psi^*(x) \psi(x)} ds \right| \tag{7}$$

where ds is a path element on the straight line which connects the points a and b.

Assume now that we are working in a D-dimensional base space, such that the wave function of a particle which is found at some energy level is given by: $\psi(x_1, ..., x_D) = \prod_{i=1}^{D} \psi_{n_i}(x_i)$, where the i-index labels the coordinates, and the n_i-index indicates in which eigenstate the particle is found (according to zero locus of ψ), over the i-coordinate (i.e. we assume separability of the variables). The base space is thus being divided naturally by the distribution

4

probability, defined on the base space, into $N \equiv \prod_{i=1}^{D} n_i$ parts (induced by the zero locus of ψ). One should thus consider the particle as being situated in any of the N parts, i.e. the location of the particle is now being represented by an N-dimensional vector. Therefore, one considers the particle as situated at one spatial point, given in N-dimensional space in its own world, but in D-dimensions in the outside world.

Figure 1. Two points a , b in the particles' world, while the particle is found in a two dimensional infinite square well potential at the energy level $n = 2$, $m = 3$. So $N = 6$ (since there are 6 cells) and $D = 2$ and therefore in this case its location will be defined by a 6 × 2 matrix

The full representation of the location of the particle thus requires an $N \times D$ - matrix, where each row represents a domain. The distance formula will thus measure the distance between two points in N-dimensional space, treated as a vector space, i.e., weighted in a Pythagorean way, and inside each domain as in eq.(7). The quantum distance formula now takes the following form:

$$d_N(a,b) = \sqrt{\sum_{i=1}^{N} \left(\frac{1}{Vol(v_i)} \left| \int_{a_i}^{b_i} \frac{1}{\psi^*(x)\psi(x)} ds_i \right| \right)^2} \tag{8}$$

5

where the i-index runs over the domains (i.e. over the rows), and by $Vol(v_i)$ one means:

$$Vol(v_i) = Vol(v) \int_{v_i} \psi^*(x)\psi(x)dx \tag{9}$$

In order that eq.(8) would be applied also to spaces with infinite volume we suggest to replace the definition of $Vol(v)$. We choose either to define $Vol(v)$ as the classical volume[3] or to define $\frac{1}{Vol(v)}$ as the square of the normalization pre-factor appears in the eigenstate (up to an arbitrary scale).

In cases where the particle state is given by a superposition of eigenstates, one has to find the $(D-1)-$ subspace on which the wave function vanishes. The space is thereby naturally separated into parts. As explained, the particle is represented in each of these parts. The $D-$ dimensional volume of each part should also be evaluated, using eq.(8). However, the main difference is that in this case the metric evolves with time. Each eigenstate in the superposition has its own time evolution, therefore the probability distribution $|\psi|^2$ also depends on time (whereas there is no such time-dependence in the case in which the particle is found in an eigenstate).

In the following section we apply the quantum distance formula eq.(8) to several examples and discuss some of the implications of the results.

---

[3]For example: for harmonic oscillator one can take the volume as based on the amplitude of the corresponding energy level, or for the hydrogen atom one can take the appropriate Bohr radius (see examples 3.3, 3.4).

3 Examples

3.1 One Dimensional - Infinite Square Well Potential

The potential is defined as:

$$U(x) = \begin{cases} 0 & 0 \leq x \leq L \\ \infty & \text{elsewhere} \end{cases} \tag{10}$$

So, the eigenstates are: $\psi_n(x) = \sqrt{\frac{2}{L}} \sin\left(\frac{n\pi x}{L}\right)$ and $Vol(v) = L$. The distances at the first energy level are therefore,

$$d_1(a,b) = \frac{1}{L} \left| \int_a^b \frac{1}{\frac{2}{L}\sin^2\left(\frac{\pi x}{L}\right)} dx \right| = \frac{L}{2\pi} \left| \cot\left(\frac{\pi b}{L}\right) - \cot\left(\frac{\pi a}{L}\right) \right| \tag{11}$$
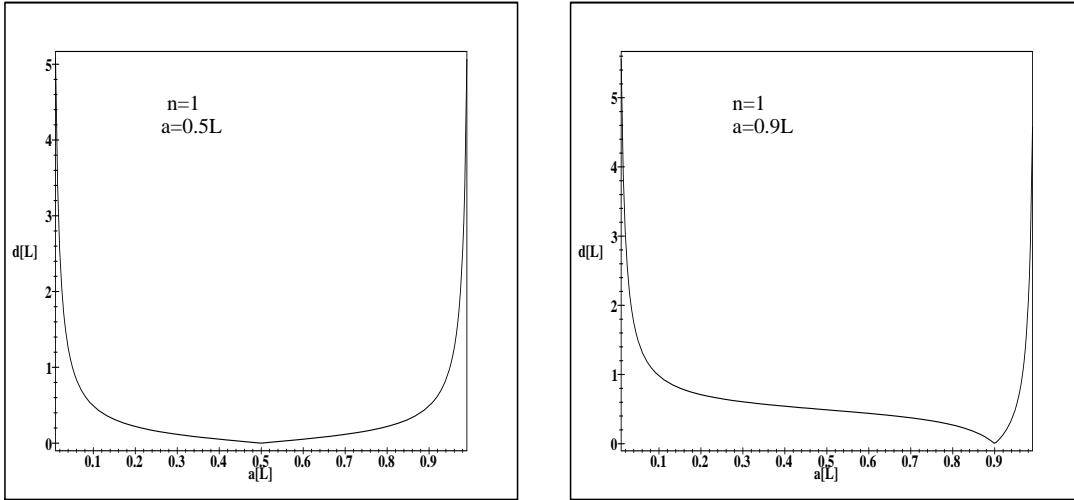

Figure 2.The distances in an infinite square well potential, at the first energy level,

as seen from the points a=0.5L and a=0.9L

The distance at the nth- level then becomes:

$$d_n(a,b) = \frac{n}{L} \sqrt{\sum_{i=1}^n \left| \int_{a_i}^{b_i} \frac{1}{\frac{2}{L}\sin^2\left(\frac{n\pi x}{L}\right)} dx \right|^2} = \frac{L}{2\pi} \sqrt{\sum_{i=1}^n \left| \cot\left(\frac{n\pi b_i}{L}\right) - \cot\left(\frac{n\pi a_i}{L}\right) \right|^2} \tag{12}$$

where $a_i, b_i \in \left(\frac{(i-1)}{n}L, \frac{i}{n}L\right)$ and $i = 1, 2, ..., n$.

3.2 Two Dimensional - Infinite Square Well Potential

The potential is defined as follows:

$$U(x,y) = \begin{cases} 0 & 0 \leq x \leq L \\ 0 & 0 \leq y \leq L \\ \infty & \text{else where} \end{cases} \tag{13}$$

The eigenstates are then: $\psi_{n,m}(x) = \frac{2}{L} \sin\left(\frac{n\pi x}{L}\right) \sin\left(\frac{m\pi y}{L}\right)$

The distance formula then becomes:

$$d_{(n,m)}(a,b) = \frac{nm}{L^2} \sqrt{\sum_{i=1}^{nm} \left| \int_{a_i}^{b_i} \frac{1}{\frac{4}{L^2} \sin^2\left(\frac{n\pi x}{L}\right) \sin^2\left(\frac{m\pi y}{L}\right)} ds_i \right|^2} \tag{14}$$

where the $i-$index stands for the $i^{th}-$zone (where there are nm zones), and a_i , b_i are respectively the initial and final points in the $i^{th}-$zone.

3.3 One Dimensional Harmonic Oscillator

The potential is:

$$U(x) = \frac{1}{2}m\omega^2 x^2 \tag{15}$$

The eigenfunctions are *Hermite polinomials*:

$$\phi_n(x) = \left[\frac{1}{2^n n!} \left(\frac{\hbar}{m\omega}\right)^n \right]^{1/2} \left(\frac{m\omega}{\pi\hbar}\right)^{1/4} \left[\frac{m\omega}{\hbar} x - \frac{d}{dx} \right]^n e^{-\frac{1}{2}\frac{m\omega}{\hbar}x^2} \tag{16}$$

so that the 0th - eigenstate is:

$$\phi_0(x) = \left(\frac{m\omega}{\pi\hbar}\right)^{1/4} e^{-\frac{1}{2}\frac{m\omega}{\hbar}x^2} \tag{17}$$

As the volume we take the "classical" amplitude A_0 at this energy level which can be found by equating the energy of this quantum level with the potential energy - i.e. $\frac{1}{2}m\omega^2 A_0^2 = \frac{1}{2}\hbar\omega$.

Thus, the "classical" amplitude at this energy level is $A_0 = \sqrt{\frac{\hbar}{m\omega}}$. Therefore the distance at the 0th $energy - level$ is:

$$d_0\left(a,b\right) = \int_a^b \sqrt{\pi} e^{\frac{m\omega}{\hbar}x^2} dx = \left(\frac{\pi}{2}\sqrt{\frac{\hbar}{m\omega}}\right) \left|\left[erf\left(i\sqrt{\frac{m\omega}{\hbar}}b\right) - erf\left(i\sqrt{\frac{m\omega}{\hbar}}a\right)\right]\right| = \quad (18)$$

$$\frac{\pi}{2}A_0\left|\left[erf\left(i\,\frac{b}{A_0}\right) - erf\left(i\,\frac{a}{A_0}\right)\right]\right|$$

For higher energy levels one has to consider the "classical" amplitudes $A_n = A_0\sqrt{2n+1}$ as the total volume - $Vol\left(v\right)$. One has then to find the $n-$roots of the $n^{th}-$Hermite polynomial. Denote the $n-$roots by $\left(\alpha_1, ..., \alpha_n\right)$. The partial volumes $Vol\left(v_i\right)$ in eq.(8) become $Vol\left(v_i\right) = Vol\left(v\right)\int_{\alpha_{i-1}}^{\alpha_i}\left|\phi_n\left(x\right)\right|^2 dx$ where $\alpha_0 = -\infty$ and $\alpha_{n+1} = \infty$. One can now make use of eq.(8) to find the distances at higher energy levels. Evaluating the distance associated with the harmonic oscillator at higher dimensions is a straightforward calculation, making use of eq.(8).
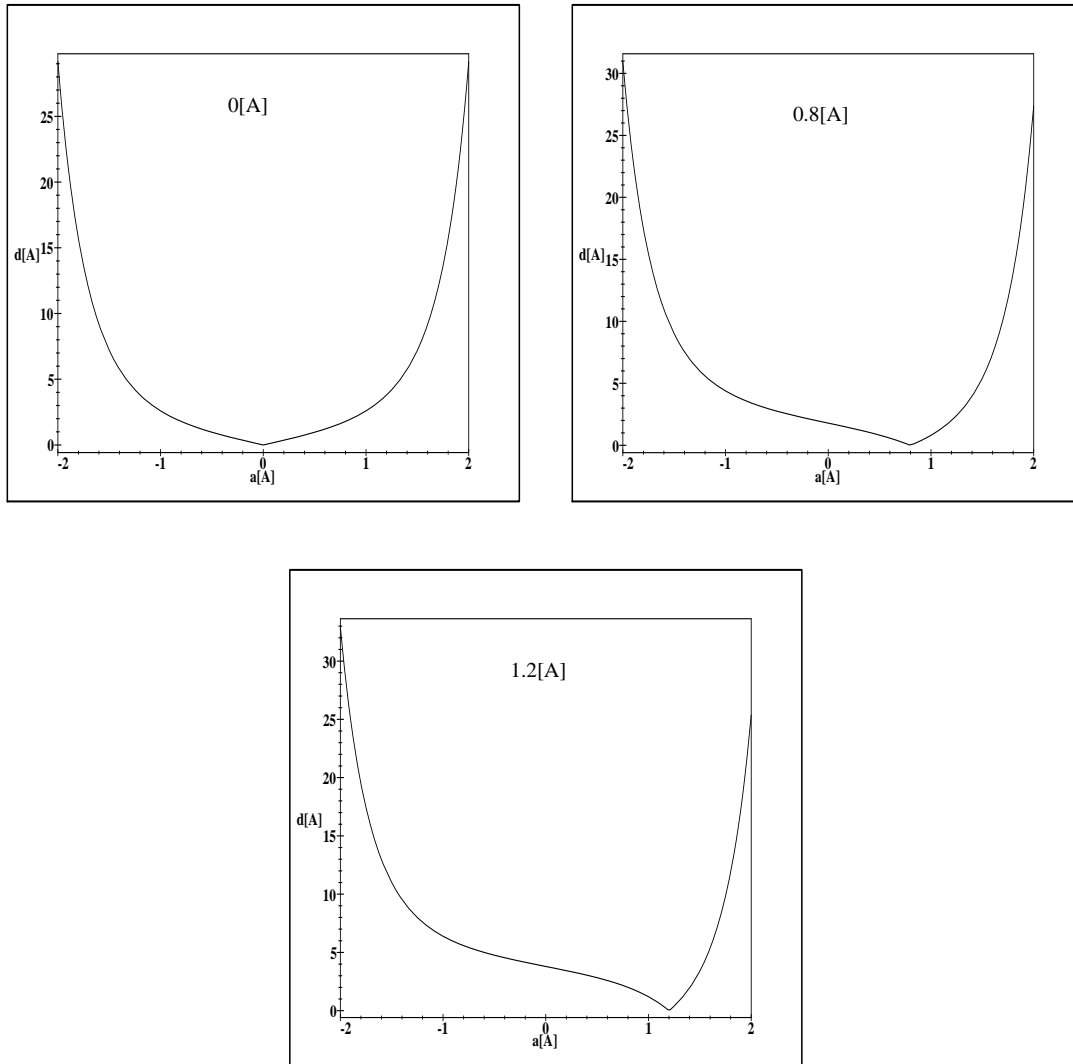
Figure 3.The distances in a one dimensional harmonic oscillator, at the n=0 energy level, as seen from the points a=0[A], a=0.8[A] and a=1.2[A]

3.4 The Hydrogen Atom

The potential is given by:

$$V(r) = -\frac{e^2}{r} \tag{19}$$

For simplicity, we compute the distances, as seen in the electron's world, in the $1s$ and the $2s$ levels and take only distances between two points situated on the same radius vector. The $1s$ and the $2s$ eigenstate are:

$$
\begin{aligned}
1s: & \quad \phi_{1s} = \frac{1}{\sqrt{\pi a_0^3}}\, e^{-r/a_0} \\
2s: & \quad \phi_{2s} = \frac{1}{\sqrt{8\pi a_0^3}}\left(1 - \frac{r}{2a_0}\right) e^{-r/(2a_0)}
\end{aligned}
\tag{20}
$$

where a_0 is the Bohr radius. In the $1s$ case we define $Vol\,(v) = \frac{4\pi}{3} a_0^3$ and the distance between two points along the same radius vector is:

$$
d_{1s}\,(r_1, r_2) = \frac{3}{8}\left|e^{2r_2/a_0} - e^{2r_1/a_0}\right| a_0
\tag{21}
$$

In the $2s$ case one sees that the eigenstate vanishes at $r = 2a_0$ and the space is thereby naturally split into two domains: $r < 2a_0$ and $r > 2a_0$. This means that according to our definition the electron's location is represented in both sectors -

i.e. $\{\,(x_1, x_2)$ where $0 \le x_1 < 2a_0$ and $2a_0 < x_2 < \infty\,\}$. The inner and outer volumes, appearing in eq.(8), will be:

$$
\begin{aligned}
Vol\,(v_1) &= \frac{4\pi}{3}(R_2)^3 \int_0^{2a_0} |\phi_{2s}\,(r)|^2\, 4\pi r^2 dr = \frac{256}{3}\pi a_0^3 \left[1 - \frac{7}{e^2}\right] \\
Vol\,(v_2) &= \frac{4\pi}{3}(R_2)^3 \int_{2a_0}^{\infty} |\phi_{2s}\,(r)|^2\, 4\pi r^2 dr = \frac{256}{3}\pi a_0^3 \frac{7}{e^2}
\end{aligned}
\tag{22}
$$

where we make use of the Bohr radius of the $n-th$ level, defined as: $R_n = n^2 a_0$. One can now make use of eq.(8) and the distance between the point (x_1, x_2) and the point (x_1', x_2') becomes:

$$
d_{(x_1, x_2)}^{(x_1', x_2')} = \sqrt{\left(\frac{1}{Vol\,(v_1)}\left|\int_{x_1}^{x_1'} |\phi_{2s}|^{-2}\, dr\right|\right)^2 + \left(\frac{1}{Vol\,(v_2)}\left|\int_{x_2}^{x_2'} |\phi_{2s}|^{-2}\, dr\right|\right)^2}
\tag{23}
$$

4   Discussion and Conclusions

It is worth reviewing the distances evaluated in the infinite square well potential, as compared to those reached in the harmonic oscillator case. While in an infinite square well potential the distances at each energy level are independent of both the mass of the particle and $\hbar$, in the harmonic oscillator the distances depend on the mass of the particle and on the oscillator

11

frequency - and also on $\hbar$. The fact that $\hbar$ does not appear in the distance formula in the infinite square well potential case might hint at one of the following:

1. Such potentials do not really exist in nature, but can be constructed as a limiting case of a superposition of harmonic oscillators.[4] In that case the walls of the infinite square well potential should be associated with the "classical amplitudes" of the harmonic oscillator (i.e. $A_i = \sqrt{\frac{\hbar}{m_i w_i}}$ where the $i-$ index label the harmonic oscillators - see also eq.(18)).

2. On the other hand, if infinite square well potentials do exist in nature, then a spectrum of mw should be induced, due to our former reasoning.

As one can see the distance between any point x for which $\psi(x) \neq 0$ and a point y for which $\psi(y) = 0$ becomes infinite. The less the probability distribution in the outside world, the larger the distances in the particle world at that neighborhood, i.e. the metric is 'stretched'. In other words, while $|\psi(x)|^2$ is the probability distribution as defined over the space in the outside world, $\frac{1}{|\psi(x)|^2}$ is the distribution of points as seen within the particle world. This can conceptually be understood from the similarity to the W.K.B. interpretation. In the W.K.B approximation $\frac{1}{|\psi(x)|^2}$ is interpreted as the average velocity of the particle in space. The velocity is however responsible for the particle's translation in space. Thus, the velocity can be interpreted as a particle's passage through a density of points in space. The larger the velocity, the larger the spatial point-density. The differential element ds in the quantum distance formula should therefore be associated with time, for the quantum formula to have a metrical meaning. Thus, in the particle world, "farther" means (according to the W.K.B approximation) a longer time in the spectator's outside world. That is to say that the spectator would have to wait a long time until he might find the particle at a place with a low probability density.[5]

---

[4]There should be an infinite of them, since the energy spectrum of the infinite square well potential is proportional to n^2 while for the harmonic oscillator it is proportional to n.

[5]Note that the interplay between time and distance elements already occurs when, one compares the metric outside and inside a black hole. It seems not to be an accident, since in both cases we have a compact space which is disconnected from the observer.

Another fact one should be aware of is that the natural splitting induced by the probability distribution $|\psi(x)|^2$ over space, in the spectator world, is being translated in the particle world into the fixing of the dimension of the space. The above conclusions can be observed in all the examples given. In this manner, the higher the energy level, the higher is the space dimensionality in the particle world.

One might ask - is the metric (in the particle world) detectable?! We might claim that any measuring process would insert a new interaction potential into the Hamiltonian, and that this would modify the eigenstates and thereby change the metric. Moreover, due to the effect of the turning on of that interaction potential, the change in the metric would be time dependent. This is just one more case in Quantum Mechanics, in which the measuring process affects the system which is being measured. However the question is how drastic is that effect. We claim that it might sometime be quite drastic, if the new eigenstate of the system and of the measurement device be changed in such a way that the quantum level is modified. This would cause a topological change - i.e. a change in the dimension (due to the change in the number of zones, as defined by the probability distribution).[6] However, if the energy level is slightly shifted - i.e. in such a way that the subspace on which $\psi(x) = 0$ changes smoothly, then the change in the metric will be a minor one.[7]

---

[6]It is well known that in the two slits experiment, if one tries to detect the "geodesics (i.e the metric)" a change in the topology is induced.

[7]The same effect exists classically - since in any process of measuring a metric, at the end, the information should be sent back to the observer. However, the information is being carried by some particle which has some energy. Thus, T_{00} is reduced and with it the metric is changed. If the observer himself wants to be part of the metric, then by that he is changing the T_{00} and with it the endowed metric.

Acknowledgment

References

[1] A. Connes, Noncommutative geometry, Academic Press Inc. 1994.

[2] I.M. Gelfand and M.A. Naimark, Izv. Akad. Nauk SSSR 12 (1948), 445-480; MR 10, 199.

[3] A. Connes, J. Lott, Particle Models and Noncommutative Geometry, Nucl. Phys. B (Proc. Suppl.) B18 (1990) 29.

A. Connes, Noncommutative Geometry and Reality, J. Math. Phys. 36 (1995) 6194-6231.

A. Connes, Gravity Coupled with Matter and The Foundation of Noncommutative Geometry, Commun. Math. Phys. 182 (1996) 155-176. hep-th/9603053

A. H. Chamseddine, A. Connes, The Spectral Action Principle, hep-th/9606001

A. H. Chamseddine, A. Connes, A Universal Action Formula, hep-th/9606056

B. Iochum, D. Kastler, T. Sch$\ddot{u}$cker, On the Universal Chamseddine-Connes Action ..., hep-th/9607158

D. Kastler, The Dirac Operator and Gravitation, Commun. Math. Phys. 166 (1995) 633-643.

G. Landi, C. Rovelli, General Relativity in Terms of Dirac Eigenvalues, gr-qc/9612034

C. P. Martin, J. M. Gracia-Bondia, J. C. Varilly, The Standard Model as a Noncommutative geometry: The Low Energy Regime hep-th/9605001.

H. Figueroa, J.M. Gracia-Bondia, F. Lizzi, J.C. Varilly, A Nonperturbative Form of the Spectral Action Principle in Noncommutative Geometry. hep-th/9701179

F. Lizzi, G. Mangano, G. Miele, G. Sparano, Constraints on Unified Gauge Theories from Noncommutative Geometry, Mod. Phys. Lett. A11 (1996) 2561-2572, hep-th/9603095.